第 24 卷 第 3 期
2007 年 9 月

建筑科学与工程学报
Journal of Architecture and Civil Engineering

Vol. 24 No. 3
Sept. 2007

文章编号:1673-2049(2007)03-0036-07

圆底球面厚扁壳的位移型控制方程及一般解

黄会荣[1],郝际平[2],黄 义[3]

(1. 西安建筑科技大学 机电工程学院,陕西 西安 710055;2. 西安建筑科技大学 土木工程学院,陕西 西安 710055;3. 西安建筑科技大学 理学院,陕西 西安 710055)

摘要: 基于考虑横向剪切变形的厚壳位移型基本方程及扁壳基本假定,建立了以 5 个中面位移为 5 个独立变量的厚扁壳位移型基本方程,并由此得到厚扁球壳在正交曲线坐标及极坐标下的位移型基本方程。为了求解圆底球面厚扁壳在极坐标下的位移型基本方程,通过引入 4 个辅助位移函数,建立其解耦的控制微分方程,最后通过这 4 个辅助位移函数求出 5 个位移分量。结果表明,厚扁壳的位移型基本方程退化为厚扁球壳及薄扁球壳的位移型方程是正确的,且所推导方程具有一般性。

关键词: 厚扁球壳;圆底球面厚扁壳;位移函数;位移型基本方程;控制方程

中图分类号: TU33 **文献标志码:** A

Displacement Governing Equations and General Solution of Circular Spherical Thick Shallow Shells

HUANG Hui-rong[1], HAO Ji-ping[2], HUANG Yi[3]

(1. School of Mechanical and Electrical Engineering, Xi'an University of Architecture and Technology, Xi'an 710055, Shaanxi, China; 2. School of Civil Engineering, Xi'an University of Architecture and Technology, Xi'an 710055, Shaanxi, China; 3. School of Science, Xi'an University of Architecture and Technology, Xi'an 710055, Shaanxi, China)

Abstract: The displacement fundamental equations of the thick shallow shells concerning five independent variables, ie five middle surface displacements were established based on the displacement fundamental equations of the thick shells by transverse shearing deformation and basic hypothesis on shallow shells, displacement fundamental equations of thick shallow spherical shells in orthogonal curvilinear coordinates and in polar coordinates were obtained. Authors introduced four assistant displacement functions to solve displacement fundamental equations of circular spherical thick shallow shells, which were tenth-order differential equations with variable coefficient and set up the decoupled governing differential equations, then obtained five displacement components through four assistant displacement functions. The results show that the displacements of equations of thick shallow shells degenerate to the displacement equations of the thick shallow spherical shells and thin shallow spherical shells, which demonstrate the generality of derived equations.

Key words: thick shallow spherical shell; circular spherical thick shallow shell; displacement function; displacement fundamental equation; governing equation

---

收稿日期: 2007-06-23

基金项目: 国家自然科学基金项目(59978038)

作者简介: 黄会荣(1963-),女,陕西西安人,副教授,工学博士研究生,E-mail:610113jiejie@163.com。

0 引　言

在工程实践中，复合材料已被广泛应用。这些材料一般都具有较低的横向剪切刚度，忽略横向剪切变形的经典扁壳理论[1-7]对解决这些工程问题会出现较大的误差，鉴于此，考虑横向剪切变形的影响是十分必要的。关于考虑横向剪切变形的厚壳计算，各国已有研究只局限于厚圆柱壳位移方程的建立，笔者在文献[8]～[10]中虽然给出了一般厚壳、厚圆柱壳、厚圆锥壳及厚球壳的位移型基本方程，但没有涉及变系数的十阶微分方程的求解分析。

板壳的位移型方程在振动问题中应用非常方便，另外，板壳的连接问题也需要位移型方程。因为，在板壳的连接问题中需要建立位移连续条件，此时应用位移型方程及其解能直接给出其位移分量，因此建立和求解板壳的位移型动力（静力）方程是十分必要的。笔者在正交曲线坐标下推导出考虑横向剪切变形厚扁壳、厚扁球壳在正交曲线坐标及极坐标下的位移型基本方程，同时，通过引入 4 个辅助位移函数 $U(r,\theta)$、$\Psi(r,\theta)$、$V(r,\theta)$、$f(r,\theta)$，还推导出圆底球面厚扁壳在极坐标下的位移型控制方程，并求出 5 个位移分量 $u_r(r,\theta)$、$u_\theta(r,\theta)$、$\varphi_r(r,\theta)$、$\varphi_\theta(r,\theta)$ 和 $w(r,\theta)$。

1 基本微分方程

在文献[9]中的厚壳位移型基本方程式中引入扁壳基本假定，则厚扁壳的位移型基本方程为

$$\begin{aligned}&\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(A_2u_1)+\frac{\partial}{\partial\alpha_2}(A_1u_2)]\}+\\&\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}[(\frac{1}{R_1}+\frac{1}{R_2})w]+\frac{1-\mu}{2}\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\cdot\\&\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_2}(A_1u_1)-\frac{\partial}{\partial\alpha_1}(A_2u_2)]\}-\\&(1-\mu)\frac{1}{A_1}\frac{1}{R_2}\frac{\partial w}{\partial\alpha_1}+(1-\mu)\frac{u_1}{R_1R_2}+\\&\frac{1}{K}q_1=\frac{1}{K}\rho J_0\ddot{u}_1\end{aligned}\tag{1}$$

$$\begin{aligned}&\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(A_2u_1)+\frac{\partial}{\partial\alpha_2}(A_1u_2)]\}+\\&\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}[(\frac{1}{R_1}+\frac{1}{R_2})w]+\frac{1-\mu}{2}\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\cdot\\&\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(A_2u_2)-\frac{\partial}{\partial\alpha_2}(A_1u_1)]\}-\\&(1-\mu)\frac{1}{A_2}\frac{1}{R_1}\frac{\partial w}{\partial\alpha_2}+(1-\mu)\frac{u_1}{R_1R_2}+\\&\frac{1}{K}q_2=\frac{1}{K}\rho J_0\ddot{u}_2\end{aligned}\tag{2}$$

$$\begin{aligned}&(\frac{1}{R_1}+\frac{1}{R_2})\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(A_2u_1)+\frac{\partial}{\partial\alpha_2}(A_1u_2)]-\\&(1-\mu)\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(\frac{A_2}{R_2}u_1)+\frac{\partial}{\partial\alpha_2}(\frac{A_1}{R_1}u_2)]+\\&(\frac{1}{R_1^2}+\frac{2\nu}{R_1R_2}+\frac{1}{R_2^2})w-\frac{1}{K}\frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_1}\cdot\\&[\frac{A_2Gh}{k_\tau}(\varphi_1+\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1})]-\frac{1}{K}\frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_2}\cdot\\&[\frac{A_1Gh}{k_\tau}(\varphi_2+\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2})]-\frac{1}{K}q_n=\frac{1}{K}\rho J_0\ddot{w}\end{aligned}\tag{3}$$

$$\begin{aligned}&\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(A_2\varphi_1)+\frac{\partial}{\partial\alpha_2}(A_1\varphi_2)]\}+\\&\frac{1-\mu}{2}\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(A_2\varphi_2)-\\&\frac{\partial}{\partial\alpha_2}(A_1\varphi_1)]\}+(1-\mu)\frac{\varphi_2}{R_1R_2}-\frac{1}{D}\cdot\\&[\frac{Gh}{k_\tau}(\varphi_2+\frac{1}{A_2}+\frac{\partial w}{\partial\alpha_2})]=\frac{1}{D}\rho J_2\ddot{\varphi}_2\end{aligned}\tag{4}$$

$$\begin{aligned}&\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_1}(A_2\varphi_1)+\frac{\partial}{\partial\alpha_2}(A_1\varphi_2)]\}+\\&\frac{1-\mu}{2}\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\{\frac{1}{A_1A_2}[\frac{\partial}{\partial\alpha_2}(A_1\varphi_1)-\\&\frac{\partial}{\partial\alpha_1}(A_2\varphi_2)]\}+(1-\mu)\frac{\varphi_1}{R_1R_2}-\frac{1}{D}\cdot\\&[\frac{Gh}{k_\tau}(\varphi_1+\frac{1}{A_1}+\frac{\partial w}{\partial\alpha_1})]=\frac{1}{D}\rho J_2\ddot{\varphi}_1\end{aligned}\tag{5}$$

式中：α_1、α_2 为中面正交曲线坐标系的坐标；u_1、u_2 为中面上 $P(\alpha_1,\alpha_2)$ 的位移分量；w 为中面上 $P(\alpha_1,\alpha_2)$ 法线方向 z 的位移分量；φ_1、φ_2 为中面上 $P(\alpha_1,\alpha_2)$ 独立转角；A_1、A_2 及 R_1、R_2 分别为拉梅系数和主曲率半径；h 为厚度；q_1、q_2、q_n 为荷载；$J_0=h$；$J_2=\frac{h^3}{12}$；$K=\frac{Eh}{1-\mu^2}$；$D=\frac{Eh^3}{12(1-\mu^2)}$。此时，横向力就变为 $T_{1n}=\frac{Gh}{k_\tau}(\varphi_1+\frac{\partial w}{\partial x})$ 和 $T_{2n}=\frac{Gh}{k_\tau}(\varphi_2+\frac{\partial w}{R\partial\theta})$。

对于球壳，若令

$$\alpha_1=\varphi,\alpha_2=\theta$$

则拉梅系数为

$$A_1=R,A_2=R\sin\varphi=r$$

主曲率半径为

$$R_1=R_2=R$$

将上述变量代入式(1)～(5)，则厚扁球壳的位移型基本方程为

$$\begin{aligned}&\frac{1}{R^2}[\frac{\partial^2u_1}{\partial\varphi^2}+\frac{1-\mu}{2}\frac{1}{\sin^2\varphi}\frac{\partial^2u_1}{\partial\theta^2}+\frac{1}{\tan\varphi}\frac{\partial u_1}{\partial\varphi}-\\&(\mu+\frac{1}{\tan^2\varphi})u_1+\frac{1+\mu}{2}\frac{1}{\sin\varphi}\frac{\partial^2u_2}{\partial\varphi\partial\theta}-\\&\frac{3-\mu}{2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial u_2}{\partial\theta}]+(1+\mu)\frac{1}{R^2}\frac{\partial w}{\partial\varphi}+\end{aligned}$$

$$\frac{1}{K}q_\varphi=\frac{1}{K}\rho J_0\ddot{u}_1 \tag{6}$$

$$\frac{1}{R^2}\Big[\frac{1+\mu}{2}\frac{1}{\sin\varphi}\frac{\partial^2 u_1}{\partial\varphi\partial\theta}+\frac{3-\mu}{2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial u_1}{\partial\theta}+\frac{1-\mu}{2}\frac{\partial^2 u_2}{\partial\varphi^2}+\frac{1}{\sin^2\varphi}\frac{\partial^2 u_2}{\partial\theta^2}+\frac{1-\mu}{2}\frac{1}{\tan\varphi}\cdot\frac{\partial u_2}{\partial\varphi}+\frac{1-\mu}{2}(2-\frac{1}{\sin^2\varphi})u_2\Big]+\frac{1}{R^2\sin\varphi}\cdot(1+\mu)\frac{\partial w}{\partial\theta}+\frac{1}{K}q_\theta=\frac{1}{K}\rho J_0\ddot{u}_2 \tag{7}$$

$$\frac{1}{R^2}(1+\mu)(\frac{\partial u_1}{\partial\varphi}+\frac{u_1}{\tan\varphi}+\frac{1}{\sin\varphi}\frac{\partial u_2}{\partial\theta}+2w)-\frac{1}{KRR_0}\frac{\partial}{\partial\varphi}\Big[\frac{R_0Gh}{k_\tau}(\varphi_1+\frac{1}{R}\frac{\partial w}{\partial\varphi})\Big]-\frac{1}{KR_0}\cdot\frac{Gh}{k_\tau}\frac{\partial}{\partial\theta}(\varphi_2+\frac{1}{R_0}\frac{\partial w}{\partial\theta})-\frac{1}{K}q_z=\frac{1}{K}\rho J_0\ddot{w} \tag{8}$$

$$\frac{1}{R^2}\Big[\frac{1+\mu}{2}\frac{1}{\sin\varphi}\frac{\partial^2\varphi_1}{\partial\varphi\partial\theta}+\frac{3-\mu}{2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial\varphi_1}{\partial\theta}+\frac{1-\mu}{2}\frac{\partial^2\varphi_2}{\partial\varphi^2}+\frac{1}{\sin^2\varphi}\frac{\partial^2\varphi_2}{\partial\theta^2}+\frac{1-\mu}{2}\frac{1}{\tan\varphi}\cdot\frac{\partial\varphi_2}{\partial\varphi}+\frac{1-\mu}{2}(2-\frac{1}{\sin^2\varphi})\varphi_2\Big]-\frac{1}{D}\Big[\frac{Gh}{k_\tau}\cdot(\varphi_2+\frac{1}{R_0}\frac{\partial w}{\partial\theta})\Big]=\frac{1}{D}\rho J_2\ddot{\varphi}_2 \tag{9}$$

$$\frac{1}{R^2}\Big[\frac{\partial^2\varphi_1}{\partial\varphi^2}+\frac{1-\mu}{2}\frac{1}{\sin^2\varphi}\frac{\partial^2\varphi_1}{\partial\theta^2}+\frac{1}{\tan\varphi}\frac{\partial\varphi_1}{\partial\varphi}-(\mu+\frac{1}{\tan^2\varphi})\varphi_1+\frac{1+\mu}{2}\frac{1}{\sin\varphi}\frac{\partial^2\varphi_2}{\partial\varphi\partial\theta}-\frac{3-\mu}{2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial\varphi_2}{\partial\theta}\Big]-\frac{1}{D}\Big[\frac{Gh}{k_\tau}(\varphi_1+\frac{1}{R}\frac{\partial w}{\partial\varphi})\Big]=\frac{1}{D}\rho J_2\ddot{\varphi}_1 \tag{10}$$

其实，厚扁球壳的位移型基本方程式(6)～(10)可以通过文献[9]中的厚球壳位移型基本方程式或文献[10]中的厚球壳位移型基本方程式及扁壳基本假定得到。

将厚扁球壳的位移型基本方程式(9)、(10)中的剪力项$\frac{Gh}{k_\tau}(\varphi_1+\frac{1}{R}\frac{\partial w}{\partial\varphi})$、$\frac{Gh}{k_\tau}(\varphi_2+\frac{1}{R_0}\frac{\partial w}{\partial\theta})$代入公式(6)～(8)中，可以得到相应的薄扁球壳的位移型方程，该方程与文献[1]、[2]中所对应的薄扁球壳方程相同。

若取惯性项为0，方程式(6)～(10)就变成厚扁球壳的位移型静力方程，这样一个变系数十阶微分方程求解十分困难，下面就讨论以极坐标表示的圆底球面厚扁壳位移型控制方程。

2 控制方程

在文献[9]中的中面应变表达式式(3)～(5)中引入扁壳的基本假定，对圆底球面扁壳有：$A_1\mathrm{d}\alpha_1=\mathrm{d}s_1=\mathrm{d}r$，$A_2=r$，$A_2\mathrm{d}\alpha_2=r\mathrm{d}\theta$，$A_1=1$，且$\frac{1}{R_1R_2}\approx0$；并记$R_1=R_2=R$；$u\to u_r$，$u=u_\theta$；$\varphi=\varphi_r$，$\psi=\varphi_\theta$；$q_1\to q_r$，$q_2\to q_\theta$；$N_1\to Q_r$，$N_2\to Q_\theta$，$T_{11}\to T_{rr}$，$M_1\to M_r$，$T_{22}\to T_{\theta\theta}$，$M_2\to M_\theta$，$T_{12}\to T_{r\theta}$，$H=M_{12}=M_{r\theta}$，则得中面薄膜应变分量表达式、中面弯曲应变分量表达式及中面横向剪切应变分量表达式为

$$\left.\begin{aligned}
&\varepsilon_{11}=\frac{\partial u_r}{\partial r}+\frac{w}{R}\\
&\varepsilon_{22}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{w}{R_2}\\
&\omega=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\\
&\kappa_1=\frac{\partial\varphi_r}{\partial r}\\
&\kappa_2=\frac{1}{r}\frac{\partial\varphi_\theta}{\partial\theta}+\frac{\varphi_r}{r}\\
&\tau=\tau_{12}+\tau_{21}=\frac{1}{r}\frac{\partial\varphi_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{\varphi_\theta}{r}\\
&\varepsilon_{13}=\varphi_r+\frac{\partial w}{\partial r}\\
&\varepsilon_{23}=\varphi_\theta+\frac{1}{r}\frac{\partial w}{\partial\theta}
\end{aligned}\right\} \tag{11}$$

采用文献[9]中的简化本构关系式(6)，并由式(11)得出其对应的内力为

$$\left.\begin{aligned}
&T_{rr}=K\Big[\frac{\partial u_r}{\partial r}+\frac{w}{R}+\mu(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{w}{R})\Big]\\
&T_{\theta\theta}=K\Big[\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{w}{R}+\mu(\frac{\partial u_r}{\partial r}+\frac{w}{R})\Big]\\
&T_{r\theta}=K\,\frac{1-\mu}{2}(\frac{1}{r}+\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r})\\
&M_r=D\Big[\frac{\partial\varphi_r}{\partial r}+\mu(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}-\frac{\varphi_r}{r})\Big]\\
&M_\theta=D(\frac{1}{r}\frac{\partial\varphi_\theta}{\partial\theta}+\frac{\varphi_r}{r}+\mu\frac{\partial\varphi_r}{\partial r})\\
&M_{r\theta}=D\,\frac{1-\mu}{2}(\frac{1}{r}\frac{\partial\varphi_r}{\partial\theta}+\frac{\partial\varphi_\theta}{\partial r}-\frac{\varphi_\theta}{r})\\
&Q_r=\frac{Gh}{k_\tau}(\varphi_r+\frac{\partial w}{\partial r})\\
&Q_\theta=\frac{Gh}{k_\tau}(\varphi_\theta+\frac{1}{r}\frac{\partial w}{\partial\theta})
\end{aligned}\right\} \tag{12}$$

由文献[9]中的方程式(9)可得出圆底球面中厚扁壳的位移型方程为

$$\frac{\partial^2u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}u_r+\frac{1-\mu}{2}\frac{1}{r^2}\frac{\partial^2u_r}{\partial\theta^2}+\frac{1+\mu}{2}\frac{1}{r}\frac{\partial^2u_\theta}{\partial r\partial\theta}-\frac{3-\mu}{2}\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+$$

$$\frac{1+\mu}{R}\frac{\partial w}{\partial r}=-\frac{q_\lambda}{K} \tag{13}$$

$$\frac{1-\mu}{2}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{1}{r^2}u_\theta\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{1+\mu}{2}\frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{3-\mu}{2}\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1+\mu}{R}\cdot\frac{1}{r}\frac{\partial w}{\partial\theta}=-\frac{q_\theta}{K} \tag{14}$$

$$-\frac{Gh}{k_\tau}\left(\frac{\partial\varphi_r}{\partial r}+\frac{1}{r}\varphi_r+\frac{1}{r}\frac{\partial\varphi_\theta}{\partial\theta}+\nabla^2 w\right)+\frac{1+\mu}{R}K\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{2}{R}w\right)=q_n+\rho J_0\ddot{w} \tag{15}$$

$$\frac{\partial^2\varphi_r}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_r}{\partial r}-\frac{1}{r^2}\varphi_r+\frac{1-\mu}{2}\frac{1}{r^2}\frac{\partial^2\varphi_r}{\partial\theta^2}+\frac{1+\mu}{2}\frac{1}{r}\frac{\partial^2\varphi_\theta}{\partial r\partial\theta}-\frac{3-\mu}{2}\frac{1}{r^2}\frac{\partial\varphi_\theta}{\partial\theta}-\frac{Gh}{k_\tau D}\cdot\left(\varphi_r+\frac{\partial w}{\partial r}\right)=\frac{1}{D}\rho J_2\ddot{\varphi}_r \tag{16}$$

$$\frac{1-\mu}{2}\left(\frac{\partial^2\varphi_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_\theta}{\partial r}-\frac{\varphi_\theta}{r^2}\right)+\frac{1}{r^2}\frac{\partial^2\varphi_\theta}{\partial\theta^2}+\frac{1+\mu}{2}\frac{1}{r}\frac{\partial^2\varphi_r}{\partial r\partial\theta}+\frac{3-\mu}{2}\frac{1}{r^2}\frac{\partial\varphi_r}{\partial\theta}-\frac{Gh}{k_\tau D}\cdot\left(\varphi_\theta+\frac{1}{r}\frac{\partial w}{\partial\theta}\right)=\frac{1}{D}\rho J_2\ddot{\varphi}_\theta \tag{17}$$

式中：$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$。

方程式(13)～(17)即为圆底球面中厚扁壳的位移型方程组，该方程组求解十分困难，为此应对方程式(13)～(17)予以简化。令

$$u_r=\frac{\partial U}{\partial r}-r\Psi,\ u_\theta=\frac{1}{r}\frac{\partial U}{\partial\theta} \tag{18}$$

$$\varphi_r=\frac{\partial V}{\partial r}-rf,\ \varphi_\theta=\frac{1}{r}\frac{\partial V}{\partial\theta} \tag{19}$$

将式(18)代入式(13)、(14)中，分别得

$$\frac{\partial}{\partial r}\left(\nabla^2 U+\frac{1+\mu}{R}w\right)-\left(r\frac{\partial^2\Psi}{\partial r^2}+3\frac{\partial\Psi}{\partial r}+\frac{1+\mu}{2}\frac{1}{r}\frac{\partial^2\Psi}{\partial\theta^2}\right)+\frac{q_r}{K}=0 \tag{20}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(\nabla^2 U+\frac{1+\mu}{R}w-\frac{1+\mu}{2}r\frac{\partial\Psi}{\partial r}-2\Psi+\frac{q_\theta}{K}\right)=0 \tag{21}$$

对式(21)两边关于θ积分得

$$\nabla^2 U+\frac{1+\mu}{R}w-\frac{1+\mu}{2}r\frac{\partial\Psi}{\partial r}-2\Psi+\frac{q_\theta}{K}=c(r) \tag{22}$$

式中：$c(r)$为任意积分函数。通常令$c(r)=0$可不失去一般性，于是式(22)变为

$$\nabla^2 U+\frac{1+\mu}{R}w-\frac{1+\mu}{2}r\frac{\partial\Psi}{\partial r}-2\Psi+\frac{q_\theta}{K}=0 \tag{23}$$

将式(23)对r进行求导，并将所得结果与式(20)相减，可得

$$\nabla^2\Psi-\frac{2}{1-\mu}\frac{1}{r}\left(\frac{q_r}{K}-\frac{1}{K}\frac{\partial q_\theta}{\partial r}\right)=0 \tag{24}$$

将式(19)代入式(16)、(17)中，分别得

$$\frac{\partial}{\partial r}\left[\nabla^2 V-\frac{Gh}{k_\tau D}(V+w)-\frac{1}{D}\rho J_2\ddot{V}\right]-\left(r\frac{\partial^2 f}{\partial r^2}+3\frac{\partial f}{\partial r}+\frac{1-\mu}{2}\frac{1}{r}\frac{\partial^2 f}{\partial\theta^2}-\frac{Gh}{k_\tau D}rf-\frac{1}{D}\rho J_2 r\ddot{f}\right)=0 \tag{25}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left[\nabla^2 V-\frac{1+\mu}{2}r\frac{\partial f}{\partial r}-2f-\frac{Gh}{k_\tau D}(V+w)-\frac{1}{D}\rho J_2\ddot{V}\right]=0 \tag{26}$$

对式(26)关于θ积分，同样取任意积分函数为0，则有

$$\nabla^2 V-\frac{Gh}{k_\tau D}(V+w)-\frac{1}{D}\rho J_2\ddot{V}-\frac{1+\mu}{2}r\frac{\partial f}{\partial r}-2f=0 \tag{27}$$

将式(27)对r进行求导，并将所得结果与式(25)相减，可得

$$\nabla^2 f-\frac{Gh}{k_\tau D}\frac{2}{1-\mu}f-\frac{2}{1-\mu}\frac{1}{D}\rho J_2\ddot{f}=0 \tag{28}$$

经检验可知，式(13)、(14)、(16)、(17)已满足以上的关系，尚余式(15)未满足，为此，将式(18)、(19)代入式(15)中，可得

$$\frac{Gh}{k_\tau}\left(\nabla^2 V+\nabla^2 w-r\frac{\partial f}{\partial r}-2f\right)-K(1+\mu)\frac{1}{R}\cdot\left(\nabla^2 U-r\frac{\partial\Psi}{\partial r}-2\Psi+\frac{2}{R}w\right)=\rho J_0\ddot{w}+q_n \tag{29}$$

至此，已将求解以u_r、u_θ、w、φ_r及φ_θ为未知量的式(13)～(17)转化为求解以U、V、w、Ψ及f为未知量的式(23)、(24)、(27)～(29)的问题。以下讨论如何对式(27)～(29)进行进一步的简化。

$$a_t=1+\frac{k_\tau\rho J_2}{Gh}\frac{\mathrm{d}^2}{\mathrm{d}t^2} \tag{30}$$

则方程式(27)、(28)分别变为

$$\nabla^2 V-\frac{Gh}{k_\tau D}a_t V-\frac{Gh}{k_\tau D}w-\frac{1+\mu}{2}r\frac{\partial f}{\partial r}-2f=0 \tag{31}$$

$$\nabla^2 f-\frac{2}{1-\mu}\frac{Gh}{k_\tau D}a_t f=0 \tag{32}$$

由式(23)可得

$$\nabla^2 U=-\frac{1+\mu}{R}w+\frac{1+\mu}{2}r\frac{\partial\Psi}{\partial r}+2\Psi-\frac{1}{K}q_\theta \tag{33}$$

将式(33)代入式(29)可得

$$\frac{Gh}{k_\tau D}(\nabla^2 V+\nabla^2 w-r\frac{\partial f}{\partial r}-2f)-\frac{K(1-\mu^2)}{D}\frac{1}{R}\cdot[\frac{w}{R}-\frac{1}{2}r\frac{\partial \Psi}{\partial r}-\frac{1}{K(1-\mu)}p_\theta]=\rho J_0\ddot{w}+q_n \quad (34)$$

将式(34)对时间 t 求导 2 次后再乘以$\frac{k_\tau \rho J_2}{Gh}$,并与式(34)相加,同时应用式(30),可得

$$\frac{Gh}{k_\tau D}[a_t\nabla^2 V+a_t\nabla^2 w-a_t(r\frac{\partial f}{\partial r}+2f)]-\frac{K(1-\mu^2)}{D}\frac{1}{R}[a_t\frac{w}{R}-a_t\frac{1}{2}r\frac{\partial \Psi}{\partial r}-\frac{1}{K(1-\mu)}a_t q_\theta]=a_t(\rho J_0\ddot{w}+q_n) \quad (35)$$

由式(31)可得

$$\nabla^4 V-\frac{Gh}{k_\tau D}a_t\nabla^2 V-\frac{Gh}{k_\tau D}\nabla^2 w-\nabla^2(\frac{1+\mu}{2}r\frac{\partial f}{\partial r}+2f)=0 \quad (36)$$

由式(36)可得

$$\frac{Gh}{k_\tau D}a_t\nabla^2 V=\nabla^4 V-\frac{Gh}{k_\tau D}\nabla^2 w-\nabla^2(\frac{1+\mu}{2}r\frac{\partial f}{\partial r}+2f) \quad (37)$$

将式(37)代入式(35)可得

$$\frac{Gh}{k_\tau D}[a_t\nabla^2 w-a_t(r\frac{\partial f}{\partial r}+2f)]+\nabla^4 V-\frac{Gh}{k_\tau D}\nabla^2 w-\nabla^2(\frac{1+\mu}{2}r\frac{\partial f}{\partial r}+2f)-\frac{K(1-\mu^2)}{D}\frac{1}{R}[a_t\frac{w}{R}-a_t\frac{1}{2}r\frac{\partial \Psi}{\partial r}-\frac{1}{K(1-\mu)}a_t q_\theta]=a_t(\rho J_0\ddot{w}+q_n) \quad (38)$$

由式(38)可得

$$\nabla^4 V=a_t(\rho J_0\ddot{w}+q_n)+\frac{Gh}{k_\tau D}\nabla^2 w+\nabla^2(\frac{1+\mu}{2}r\frac{\partial f}{\partial r}+2f)-\frac{Gh}{k_\tau D}a_t\nabla^2 w+\frac{Gh}{k_\tau D}a_t(r\frac{\partial f}{\partial r}+2f)+\frac{K(1-\mu^2)}{D}\frac{1}{R}\cdot[a_t\frac{w}{R}-a_t\frac{1}{2}r\frac{\partial \Psi}{\partial r}-\frac{1}{K(1-\mu)}a_t q_\theta] \quad (39)$$

再由式(34)可得

$$\frac{Gh}{k_\tau D}[\nabla^4 V+\nabla^4 w-\nabla^2(r\frac{\partial f}{\partial r}+2f)]-\frac{K(1-\mu^2)}{D}\frac{1}{R}[\frac{1}{R}\nabla^2 w-\frac{1}{2}\nabla^2(r\frac{\partial \Psi}{\partial r})-\frac{1}{K(1-\mu)}\nabla^2 q_\theta]=\nabla^2(\rho J_0\ddot{w}+q_n) \quad (40)$$

将式(39)代入式(40),可得

$$\frac{Gh}{k_\tau D}\{a_t(\rho J_0\ddot{w}+q_n)+\frac{Gh}{k_\tau D}(1-a_t)\nabla^2 w+\nabla^2(\frac{1+\mu}{2}r\frac{\partial f}{\partial r}+2f)+\frac{Gh}{k_\tau D}a_t(r\frac{\partial f}{\partial r}+2f)+\frac{K(1-\mu^2)}{D}\frac{1}{R}[a_t\frac{w}{R}-a_t\frac{1}{2}r\frac{\partial \Psi}{\partial r}-\frac{1}{K(1-\mu)}a_t q_\theta]+\nabla^4 w-\nabla^2(r\frac{\partial f}{\partial r}+2f)\}-\frac{K(1-\mu^2)}{D}\frac{1}{R}[\frac{1}{R}\nabla^2 w-\frac{1}{2}\nabla^2(r\frac{\partial \Psi}{\partial r})-\frac{1}{K(1-\mu)}\nabla^2 q_\theta]=\nabla^2(\rho J_0\ddot{w}+q_n) \quad (41)$$

对式(41)进行整理后得

$$\frac{Gh}{k_\tau D}\{a_t(\rho J_0\ddot{w}+q_n)+\frac{Gh}{k_\tau D}(1-a_t)\nabla^2 w-\frac{1-\mu}{2}\nabla^2(r\frac{\partial f}{\partial r})+\frac{Gh}{k_\tau D}a_t(r\frac{\partial f}{\partial r}+2f)+\frac{K(1-\mu^2)}{D}\frac{1}{R}[a_t\frac{w}{R}-a_t\frac{1}{2}r\frac{\partial f}{\partial r}-\frac{1}{K(1-\mu)}a_t q_\theta]+\nabla^4 w]\}-\frac{K(1-\mu^2)}{D}\cdot\frac{1}{R}[\frac{1}{R}\nabla^2 w-\frac{1}{2}\nabla^2(r\frac{\partial \Psi}{\partial r})-\frac{1}{K(1-\mu)}\nabla^2 q_\theta]=\nabla^2(\rho J_0\ddot{w}+q_n) \quad (42)$$

应用等式$\nabla^2 r\frac{\partial(\because)}{\partial r}=(r\frac{\partial}{\partial r}+2)\nabla^2(\because)$及式(32),则式(42)变为

$$\frac{Gh}{k_\tau D}\{a_t(\rho J_0\ddot{w}+q_n)+\frac{Gh}{k_\tau D}(1-a_t)\nabla^2 w-\frac{Gh}{k_\tau D}a_t(r\frac{\partial f}{\partial r}+2f)+\frac{Gh}{k_\tau D}a_t(r\frac{\partial f}{\partial r}+2f)+\frac{K(1-\mu^2)}{D}\frac{1}{R}[a_t\frac{w}{R}-a_t\frac{1}{2}r\frac{\partial \Psi}{\partial r}-\frac{1}{K(1-\mu)}a_t q_\theta]+\nabla^4 w]\}-\frac{K(1-\mu^2)}{D}\cdot\frac{1}{R}[\frac{1}{R}\nabla^2 w-\frac{1}{1-\mu}(r\frac{\partial}{\partial r}+2)\frac{1}{r}\frac{1}{K}\cdot(q_r-\frac{\partial q_\theta}{\partial r})-\frac{1}{K(1-\mu)}\nabla^2 q_\theta]=\nabla^2(\rho J_0\ddot{w}+q_n) \quad (43)$$

对式(43)等式两边作用算子∇^2,可得

$$\frac{Gh}{k_\tau D}\{a_t\nabla^2(\rho J_0\ddot{w}+q_n)+\frac{Gh}{k_\tau D}(1-a_t)\nabla^4 w+\frac{K(1-\mu^2)}{D}\frac{1}{R}[\frac{1}{R}a_t\nabla^2 w-a_t\frac{1}{2}\nabla^2\cdot(r\frac{\partial \Psi}{\partial r})-\frac{1}{K(1-\mu)}a_t\nabla^2 q_\theta]+\nabla^6 w\}-$$

$$\frac{K(1-\mu^2)}{D}\frac{1}{R}\left[\frac{1}{R}\nabla^4 w-\frac{1}{K(1-\mu)}\nabla^2\cdot\right.$$
$$(r\frac{\partial}{\partial r}+2)(\frac{1}{r}q_r-\frac{1}{r}\frac{\partial q_\theta}{\partial r})-$$
$$\left.\frac{1}{K(1-\mu)}\nabla^4 q_\theta\right]=\nabla^4(\rho J_0\ddot{w}+q_n) \tag{44}$$

因为 $\nabla^2(r\frac{\partial \Psi}{\partial r})=(r\frac{\partial}{\partial r}+2)\nabla^2\Psi$，且应用式(24)，则式(44)可变为以单一变量 $w(r,\theta)$ 表示的单一方程，即

$$\frac{Gh}{k_\tau D}\{a_t\nabla^2(\rho J_0\ddot{w}+q_n)+\frac{Gh}{k_\tau D}(1-a_t)\nabla^4 w+$$
$$\frac{K(1-\mu^2)}{D}\frac{1}{R}\left[\frac{1}{R}a_t\nabla^2 w-a_t\frac{1}{1-\mu}\frac{1}{K}\cdot\right.$$
$$(r\frac{\partial}{\partial r}+2)(\frac{q_r}{r}-\frac{1}{r}\frac{\partial q_\theta}{\partial r})-\frac{1}{K(1-\mu)}\cdot$$
$$\left.a_t q_\theta\right]+\nabla^6 w\}-\frac{K(1-\mu^2)}{D}\frac{1}{R}\left[\frac{1}{R}\nabla^4 w-\right.$$
$$\frac{1}{K(1-\mu)}\nabla^2(r\frac{\partial}{\partial r}+2)(\frac{q_r}{r}-\frac{1}{r}\frac{\partial q_\theta}{\partial r})-$$
$$\left.\frac{1}{K(1-\mu)}\nabla^4 q_\theta\right]=\nabla^4(\rho J_0\ddot{w}+q_n) \tag{45}$$

对式(45)进行整理后，可得

$$\nabla^6 w+b_1\nabla^4 w+b_2\nabla^2 w+P=0 \tag{46}$$

$$\left.\begin{aligned}
&b_1=(\frac{Gh}{k_\tau D})^2(1-a_t)-\frac{K(1-\mu^2)}{D}\frac{1}{R^2}-\rho J_0\frac{\mathrm{d}^2}{\mathrm{d}t^2}\\
&b_2=\frac{Gh}{k_\tau D}a_t\left[\rho J_0\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{K(1-\mu^2)}{D}\frac{1}{R^2}\right]\\
&P=(-\nabla^4+\frac{Gh}{k_\tau D}a_t)q_n+\frac{K(1-\mu^2)}{D}\frac{1}{R}\cdot\\
&\quad\frac{1}{K(1-\mu)}(\nabla^2-a_t)(r\frac{\partial}{\partial r}+2)(\frac{q_r}{r}-\\
&\quad\frac{1}{r}\frac{\partial q_\theta}{\partial r})+\frac{K(1-\mu^2)}{D}\frac{1}{R}\frac{1}{K(1-\mu)}\cdot\\
&\quad(\nabla^4-a_t\nabla^2)q_\theta
\end{aligned}\right\} \tag{47}$$

由式(29)可得

$$\frac{Gh}{k_\tau D}(\nabla^2 V+\nabla^2 w-r\frac{\partial f}{\partial r}-2f)-$$
$$\frac{K(1-\mu)}{D}\frac{1}{R}(\nabla^2 U-r\frac{\partial\Psi}{\partial r}-$$
$$2\Psi+\frac{2}{R}w)=\frac{1}{D}(\rho J_0\ddot{w}+q_n) \tag{48}$$

将式(23)、(31)代入式(48)，可得

$$(\frac{Gh}{k_\tau D})^2 a_t V=-\frac{Gh}{k_\tau D}\nabla^2 w-\left[(\frac{Gh}{k_\tau D})^2-\right.$$
$$\left.\frac{K(1-\mu^2)}{D}\frac{1}{R^2}\right]w+\frac{1-\mu}{2}\frac{Gh}{k_\tau D}r\frac{\partial f}{\partial r}-$$
$$\frac{K(1-\mu^2)}{D}\frac{1}{R}\frac{1}{2}r\frac{\partial\Psi}{\partial r}-$$
$$\frac{(1+\mu)}{RD}q_\theta+\frac{1}{D}(\rho J_0\ddot{w}+q_n) \tag{49}$$

方程式(23)、(24)、(32)、(46)、(49)即为简化后的控制方程组，若略去圆底球面厚扁壳方程中相应的惯性项便可得出相应静力问题的控制方程。

对圆底球面厚扁壳，首先从控制式(24)、(32)和(46)分别求出位移函数 $\Psi(r,\theta)$、$f(r,\theta)$ 及位移分量 $w(r,\theta)$，再通过控制式(23)、(49)分别求出位移函数 $U(r,\theta)$、$V(r,\theta)$，进一步通过式(19)～(21)求出 5 个位移分量 $u_r(r,\theta)$、$u_\theta(r,\theta)$、$\varphi_r(r,\theta)$、$\varphi_\theta(r,\theta)$ 和 $w(r,\theta)$。

3　结　语

将厚扁球壳位移型基本方程式(9)、(10)中的剪力项 $\frac{Gh}{k_\tau}(\varphi_1+\frac{1}{R}\frac{\partial w}{\partial\varphi})$、$\frac{Gh}{k_\tau}(\varphi_2+\frac{1}{R_0}\frac{\partial w}{\partial\theta})$ 代入式(6)～(8)中，可以得到相应的薄扁球壳的位移型方程，该方程与文献[1]、[2]中所对应的薄扁球壳方程相同，说明了式(6)～(10)的正确性。

为使厚扁球壳的变系数十阶微分方程式(13)～(17)解耦，引入 4 个辅助位移函数 $U(r,\theta)$、$\Psi(r,\theta)$、$V(r,\theta)$、$f(r,\theta)$，建立其解耦的控制微分方程，最后通过 4 个辅助位移函数求出 5 个位移分量，所求出的 5 个位移分量解具有一般性。

参考文献：

References：

[1] 曹志远. 板壳振动理论[M]. 北京：中国铁道出版社，1989.
CAO Zhi-yuan. Theory of Plates and Shells Vibration[M]. Beijing：China Railway Publishing House，1989.

[2] 黄克智. 弹性薄壳理论[M]. 北京：科学出版社，1988.
HUANG Ke-zhi. Elastic Thin-shell Theory[M]. Beijing：Science Press，1988.

[3] 诺沃日洛夫 V V. 薄壳理论[M]. 北京石油学院材料力学教研组，译. 北京：科学出版社，1959.
NOVOZHILOV V V. Thin Shell Theory[M]. Translated by Material Mechanic Teaching and Research Group of Beijing Institute of Petroleum. Beijing：Science Press，1959.

[4] 何福保，沈亚鹏. 板壳理论[M]. 西安：西安交通大学出版社，1993.
HE Fu-bao，SHEN Ya-peng. Theory of Plates and Shells[M]. Xi'an：Xi'an Jiaotong University Press，1993.

[5] CLIVE L D. Introduction to the Theory of Shells

[M]. New York: Pergamon Press, 1974.

[6] 黄会荣,郝际平.考虑弯曲内力轴对称旋转壳的非线性协调方程[J].西北大学学报:自然科学版,1999,29(5):377-380.
HUANG Hui-rong, HAO Ji-ping. The Nonlinear Coordinate Equations of Rotation Shell with Rotational Symmetry by Considering Bending Internal Forces [J]. Journal of Northwest University: Natural Science Edition, 1999, 29(5): 377-380.

[7] 黄会荣,李 革.轴对称旋转薄壳的非线性控制方程[J].西安建筑科技大学学报:自然科学版,2000,32(2):192-195.
HUANG Hui-rong, LI Ge. Nonlinear Governing Equations of the Rotational Thin Shell with Axial Symmetry[J]. Journal of Xi'an University of Architecture and Technology: Natural Science Edition, 2000, 32(2): 192-195.

[8] 黄会荣,黄 义.基于精确本构关系中厚壳振动的位移型基本方程[J].西北大学学报:自然科学版,2002,32(5):20-24.
HUANG Hui-rong, HUANG Yi. Displacemental Fundamental Equations of the Shells Vibration by Transvers Shearing Defomation[J]. Journal of Northwest University: Natural Science Edition, 2002, 32(5): 20-24.

[9] 黄会荣,黄 义.考虑横向剪切变形厚球壳的位移型动力方程[J].西安建筑科技大学学报:自然科学版,2002,34(2):170-175.
HUANG Hui-rong, HUANG Yi. Displacemental Dynamical Equations of the Spherical Shells by Transverse Shearing Deformation[J]. Journal of Xi'an University of Architecture and Technology: Natural Science Edition, 2002, 34(2): 170-175.

[10] 黄会荣,黄 义.考虑横向剪切变形厚旋转壳振动的位移型基本方程[J].空间结构,2003,10(1):3-7.
HUANG Hui-rong, HUANG Yi. Displacemental Vibration Equations of Thick Shells of Revolution Considering Transverse Shear Deformation [J]. Spatial Structures, 2003, 10(1): 3-7.